Solution to Problem 8) Our first example is a 2×2 matrix with a single (degenerate) eigen-value and a single eigen-vector. Consequently, this matrix is *not* diagonalizable.

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad |A_1 - \lambda I| = (1-\lambda)^2 = 0 \qquad \rightarrow \qquad \lambda_{1,2} = 1.$$

$$(A_1 - \lambda I)V = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \;\rightarrow\; (v_1 = \text{arbitrary},\; v_2 = 0) \;\rightarrow\; V = \begin{pmatrix} 1 \\ 0 \end{pmatrix} v_1.$$

Our second example is another 2×2 matrix, but this one has complex elements. Again, the matrix turns out to have a single (degenerate) eigen-value and a single eigen-vector. As such, this matrix is *not* diagonalizable.

$$A_2 = \begin{pmatrix} 1+\mathrm{i} & 2 \\ 0 & 1+\mathrm{i} \end{pmatrix}, \quad |A_2 - \lambda I| = (1+\mathrm{i}-\lambda)^2 = 0 \quad \rightarrow \quad \lambda_{1,2} = 1+\mathrm{i}.$$

$$(A_2 - \lambda I)V = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \;\rightarrow\; (v_1 = \text{arbitrary},\; v_2 = 0) \;\rightarrow\; V = \begin{pmatrix} 1 \\ 0 \end{pmatrix} v_1.$$

For our third example, we pick a 3×3 matrix which has two eigen-values, one being degenerate. This matrix turns out to have only two independent eigen-vectors. Once again, there are not enough eigen-vectors to diagonalize the matrix.

$$A_3 = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 2 & 0 \\ 4 & 0 & 5 \end{pmatrix}.$$

$$|A_3 - \lambda I| = (1-\lambda)(2-\lambda)(5-\lambda) + 4(2-\lambda) = (2-\lambda)(3-\lambda)^2 = 0$$

$$\rightarrow \;\; \lambda_1 = 2, \qquad \lambda_2 = \lambda_3 = 3.$$

The eigen-vector associated with λ_1 is computed as follows:

$$(A_3 - \lambda_1 I)V_1 = \begin{pmatrix} -1 & 0 & -1 \\ 3 & 0 & 0 \\ 4 & 0 & 3 \end{pmatrix}\begin{pmatrix} v_{11} \\ v_{12} \\ v_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \;\; (v_{11} = v_{13} = 0, \quad v_{12} = \text{arbitrary}) \qquad \rightarrow \qquad V_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} v_{12}.$$

The eigen-vector associated with the degenerate eigen-values λ_2 and λ_3 is found to be

$$(A_3 - \lambda_2 I)V_2 = \begin{pmatrix} -2 & 0 & -1 \\ 3 & -1 & 0 \\ 4 & 0 & 2 \end{pmatrix}\begin{pmatrix} v_{21} \\ v_{22} \\ v_{23} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \;\; (v_{22} = 3v_{21}, \quad v_{23} = -2v_{21}) \qquad \rightarrow \qquad V_2 = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} v_{21}.$$

Our fourth example is another 2×2 matrix with real entries. Unlike the previous examples, this matrix has two non-degenerate eigen-values and, consequently, two linearly independent eigen-vectors. Below, we find both eigen-values and their associated eigen-vectors, then proceed to diagonalize the matrix.

$$A_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad |A_4 - \lambda I| = \lambda^2 + 1 = 0 \quad \rightarrow \quad \lambda_{1,2} = \pm \mathrm{i}.$$

$$(A_4 - \lambda I)V = \begin{pmatrix} \mp\mathrm{i} & -1 \\ 1 & \mp\mathrm{i} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \; v_1 = \pm\mathrm{i} v_2 \;\; \rightarrow \; V_{1,2} = \begin{pmatrix} \pm\mathrm{i} \\ 1 \end{pmatrix} v_2.$$

$$\tilde{V} = \begin{pmatrix} \mathrm{i} & -\mathrm{i} \\ 1 & 1 \end{pmatrix}, \qquad \tilde{V}^{-1} = ½ \begin{pmatrix} -\mathrm{i} & 1 \\ \mathrm{i} & 1 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}.$$

Finally, the diagonalized matrix A_4 is written as follows:

$$A_4 = \tilde{V} \Lambda \tilde{V}^{-1} = ½ \begin{pmatrix} \mathrm{i} & -\mathrm{i} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \begin{pmatrix} -\mathrm{i} & 1 \\ \mathrm{i} & 1 \end{pmatrix}.$$

Our next example, once again, is a 2×2 matrix with real entries, which has a single (degenerate) eigen-value, and a single eigen-vector associated with that eigen-value. In the absence of a sufficient number of (linearly independent) eigen-vectors, the matrix cannot be diagonalized.

$$A_5 = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, \quad |A_5 - \lambda I| = -\lambda(2 - \lambda) + 1 = (\lambda - 1)^2 = 0 \quad \rightarrow \quad \lambda_{1,2} = 1.$$

$$(A_5 - \lambda I)V = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \; \rightarrow \; v_1 = v_2 = \text{arbitrary} \; \rightarrow \; V = \begin{pmatrix} 1 \\ 1 \end{pmatrix} v_1.$$

Our final example is a 3×3 matrix with complex entries and only two eigen-values. In spite of the degeneracy, however, the matrix has three independent eigen-vectors. We compute both eigen-values and all three (linearly-independent) eigen-vectors. We then proceed to diagonalize the matrix with the aid of these eigen-values and eigen-vectors.

$$A_6 = \begin{pmatrix} 1 & 0 & \mathrm{i} \\ 0 & 2 & 0 \\ -\mathrm{i} & 0 & 1 \end{pmatrix}.$$

$$|A_6 - \lambda I| = (1 - \lambda)(2 - \lambda)(1 - \lambda) + \mathrm{i}^2 (2 - \lambda) = (2 - \lambda)(\lambda^2 - 2\lambda) = 0.$$

$$\rightarrow \qquad \lambda_1 = 0, \qquad \lambda_2 = \lambda_3 = 2.$$

The eigen-vector associated with λ_1 is now determined as follows:

$$(A_6 - \lambda_1 I)V_1 = \begin{pmatrix} 1 & 0 & \mathrm{i} \\ 0 & 2 & 0 \\ -\mathrm{i} & 0 & 1 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \\ v_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \quad (v_{13} = \mathrm{i} v_{11}, \qquad v_{12} = 0) \qquad \rightarrow \qquad V_1 = \begin{pmatrix} 1 \\ 0 \\ \mathrm{i} \end{pmatrix} v_{11}.$$

The next step is to compute the eigen-vector(s) associated with the degenerate eigen-values λ_2 and λ_3, as follows:

$$(A_6 - \lambda_2 I)V_2 = \begin{pmatrix} -1 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & -1 \end{pmatrix}\begin{pmatrix} v_{21} \\ v_{22} \\ v_{23} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \ (v_{23} = -\mathrm{i}v_{21}, \quad v_{22} = \text{arbitrary}) \quad \rightarrow \quad V_2 = \begin{pmatrix} 1 \\ 0 \\ -\mathrm{i} \end{pmatrix} v_{21}, \quad V_3 = \begin{pmatrix} 1 \\ 1 \\ -\mathrm{i} \end{pmatrix} v_{22}.$$

The matrix A_6 is now diagonalized with the aid of the matrix of eigen-vectors, namely,

$$\tilde{V} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ \mathrm{i} & -\mathrm{i} & -\mathrm{i} \end{pmatrix}, \qquad \tilde{V}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 & -\mathrm{i} \\ 1 & -2 & \mathrm{i} \\ 0 & 2 & 0 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

$$A_6 = \tilde{V}\Lambda\tilde{V}^{-1} = ½\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ \mathrm{i} & -\mathrm{i} & -\mathrm{i} \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & -\mathrm{i} \\ 1 & -2 & \mathrm{i} \\ 0 & 2 & 0 \end{pmatrix}.$$

If need be, one can orthonormalize the eigen-vectors constituting the columns of $\tilde{V}$, so that the diagonalization of the Hermitian matrix A_6 is carried out via a unitary matrix.
